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ZEROS OF VECTOR FIELDS AND CHARACTERISTIC NUMBERS

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1. Introduction

Let M be a compact oriented C^∞-smooth n $(=4m)$-dimensional manifold, and X a smooth vector field on M. Suppose that the vector field X has a set S of isolated zero-order zero points. Furthermore, let us assume that we are given a vector bundle E over M and that the flow, defined by X on M, lifts to a flow on E. Let $\Phi(c(E^c))$ be a polynomial in Chern classes of the complexification E^c of E. Then in certain special cases one can define [1] a singular pair $\phi_\Phi = (\phi_E, \alpha_E)$ of forms (depending on Φ and E), where α_E has S as a set of its singular points, ϕ_E is smooth everywhere, and $\phi_E = d\alpha_E$ on $M - S$ such that the residue of this singular pair determines the Chern numbers corresponding to $\Phi(c(E^c))$, namely,

$$\Phi(c(E^c))[M] = \operatorname{Res} \phi_\Phi \,. \tag{1.1}$$

This formula can be looked at [5] as a generalization of the classical Hopf theorem, which says that the Euler characteristic of a manifold is equal to the number of isolated zero points of a vector field on that manifold—each zero point taken with appropriate sign. From this point of view formula (1.1) was derived for various cases (M a complex analytic manifold with X meromorphic and E a holomorphic vector bundle, and M a riemannian manifold with X as killing vector field which has a lift to a real vector bundle E) by Baum, Bott, Cheeger. The case where the set S of zero points of X is a collection of submanifolds has also been considered. In all these cases one can assume that Φ is an arbitrary polynomial in (1.1). The vector fields which have been considered satisfy obvious elliptic differential equations. The following question arises: can one obtain formula (1.1) under the assumption that X satisfies $DX = 0$ for some elliptic operator D? In this case the residue formula (1.1) can be derived from the general methods of Atiyah and Singer [3] but under an additional assumption that the 1-parameter group $\{\exp(tX)\}$ is a subgroup of a compact group. But this fails to be the case even for the holomorphic vector fields. Therefore it is interesting to find an analogue of (1.1) when the 1-para-

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meter group $\{\exp(tX)\}$ is not necessarily a subgroup of a compact group.

In this paper we consider a compact manifold M together with a first order elliptic transitive oriented involutive pseudogroup Γ on it. Furthermore we assume that X is a Γ-vector field with a set S of isolated zero-order zero points and that the endomorphism L_p of $T_p(M)$, defined by the Lie derivative L_X at p, has the eigenvalues $\lambda_1(p), \cdots, \lambda_n(p)$ at all points $p \in S$. Let T_c be the complexified tangent bundle $T(M)$, and $c(T_c) = \prod_{i=1}^{n} (1 + y_i)$ the formal factorization of the total Chern class. Then we can express the j-th Chern class $c_j(T_c)$ as an elementary symmetric function of the y_i's, and we define the polynomial ϕ to each Φ by the relation $\Phi(c(T_c)) = \Phi(c_1(T_c), \cdots, c_l(T_c)) = \phi(y_1, \cdots, y_n)$, $2l = n$. In this case (1.1) reads (Theorem 7.1):

$$\Phi(c(T_c))[M] = \sum 4\varepsilon_p \frac{\phi(\lambda_1(p), \cdots, \lambda_n(p))}{(\lambda_1(p) \cdots \lambda_n(p))^{1/2}}, \tag{1.2}$$

where $\varepsilon_p = \text{sign}\det(1 - e^{L_p})$.

Since Φ can not be arbitrary in formula (1.2), $\Phi(c(T_c))$ has to be the index cocycle of the (elliptic complex) resolution of the sheaf of germs of Γ-vector fields, so that the denominator $(\lambda_1(p) \cdots \lambda_n(p))^{-1/2}$ can be interpreted as a residue of certain cohomology class (Theorem 5.1); this theorem is a generalization of similar residue formula of Bott for holomorphic vector field on a complex manifold.

2. Resolution of the sheaf of Γ vector fields

Let M be an n-dimensional ($n = 2l = 4m$) C^∞-smooth manifold with a transitive k-th order pseudogroup Γ on it. For all basic definitions and fundamental properties, concerning pseudogroups and related operators, we refer to [6], [7], [9]. A vector field Y on M is called a Γ-vector field if the one-parameter group of transformations $\{\exp(tY)\}$ belongs to Γ. The vector bundle of r-jets, $r \geq 0$, of all Γ vector fields on M will be denoted by R_r. R_r is a subbundle of the bundle of r-jets of sections of the tangent bundle $T = T(M)$. There is a well defined differential operator of order k on T, namely,

$$\nabla_k : T \to J^k(T)/R_k . \tag{2.1}$$

This operator factors through $J^k(T)$ so that $\nabla_k = d_k \cdot j^k$, where j^k is the operator which takes a section into its k-jet, and d_k is a bundle map. Obviously $R_k = \ker d_k$. If the operator ∇_k is involutive we say that the pseudogroup Γ is an involutive pseudogroup. Let us denote by V_r the typical fibre of the vector bundle R_r and by G_{r+1} its structure group. The principal G_{r+1}-bundle associated with R_r will be denoted by $P_{r+1} \to M$. The group G_{r+1} being the structure group of R_r is in fact a subgroup of $GL(\dim V_r, R)$. On the other hand, G_{r+1} is isomorphic to the group of $(r + 1)$-jets of all transformations of Γ preserving a fixed point $0 \in M$. The jet projection gives a surjective map $G_r \to G_s \to 1$, $r \geq s$. We shall denote by $\pi\colon R_r \to R_s$ the jet projection, $r \geq s$. In each conjugacy class of the maximal compact subgroups of G_r, $r \geq 0$, one can choose a group H_r such that the surjective map $G_r \to G_s$, $r \geq s$, carries H_r into H_s. In fact we have the following proposition.

Proposition 2.1. *The jet projection $G_r \to G_s$ induces the isomorphism*

$$p_s^r : H_r \to H_s\,, \qquad r \geq s \geq 1\,. \tag{2.2}$$

Proof. First, let us show that the map p_1^r is injective. Suppose that there is a nonzero element $k' \in \ker p_1^r \subset H_r \subset G_r$. Then there exists $\phi \in \Gamma$, $\phi(0) = 0$, such that $j_0^r\phi = k'$, where j_0^r is the r-jet of ϕ at 0. Let us choose local coordinates $(x^1, \cdots, x^n)$ in a neighborhood U of 0. Then ϕ has the components $\phi^j = \phi^j(x^1, \cdots, x^n)$, $j = 1, \cdots, n$. If we denote by $\phi^j_{i_1 \cdots i_s}$ the partial derivative $(\partial^s\phi^j/\partial x^{i_1}\cdots\partial x^{i_s})(0)$, then k' has coordinates $(0, \delta_i^j, \phi^j_{i_1i_2}, \cdots, \phi^j_{i_1\cdots i_r})$; $i, i_1, \cdots, i_r = 1, \cdots, n$; where δ_i^j is the Kronecker symbol.

From the composition of jets it follows that $k'^2 = k' \cdot k'$ has local coordinates

$$(0, \delta_i^j, 2\phi^j_{i_1i_2}, 2\phi^j_{i_1i_2i_3} + \Sigma\phi^j_{li_2}\phi^l_{i_1i_3} + \Sigma\phi^j_{i_1l}\phi^l_{i_2i_3}, \cdots)\,.$$

By induction we get the local coordinates $(0, \delta_i^j, N\cdot\phi^j_{i_1i_2}, \cdots)$ for $k'N$.

Now we consider the image of $\ker p_1^r$ in the euclidean space of sufficiently high dimension under the mapping which sends k' to the point with coordinates $(0, \delta_i^j, \phi^j_{i_1i_2}, \cdots, \phi^j_{i_1\cdots i_r})$. This mapping being continuous and bijective guarantees that the image of $\ker p_1^r$ is closed and bounded. But this is possible only if $\phi^j_{i_1\cdots i_s} = 0$, $2 \leq s \leq r$. This proves that p_1^r is injective. Surjectivity of the map p_1^r follows from the fact that H_r is a maximal compact subgroup.

Definition 2.1. The pseudogroup Γ is oriented if the vector bundle R_r is an oriented vector bundle for $r \geq 0$.

From now on we assume that Γ is oriented. Then the structure group G_{r+1} of R_r can be reduced to the maximal compact subgroup H_r.

Moreover, by the Proposition 2.1, the bundle R_r, $r \geq 1$, has H_1 as its structure group. Let us denote by $\pi_1\colon P_{H_1} \to B_{H_1}$ the Stiefel H_1-bundle $SO(N)/SO(N-n)$ over the Grassmann manifold $SO(N)/(SO(N-n) \times H_1)$ for some fixed sufficiently large N.

From the universal bundle theorem and the above proposition, it follows that the bundles R_r and $R_0 = T$, $g_{r+1} =$ kernel of the jet projection $R_{r+1} \to R_r$, are pull backs by certain map $f\colon M \to B_{H_1}$ of the vector bundles

$$\begin{aligned} &E_r = P_{H_1} \times_{H_1} V_r\,, \qquad r \geq 0\,,\\ &A = E_0\,,\\ &K_{r+1} = P_{H_1} \times_{H_1} V_r^{r+1}\,, \quad V_r^{r+1} = \ker(V_{r+1} \to V_r)\,, \quad r \geq 0\,, \end{aligned} \tag{2.3}$$

over B_{H_1}. In other words there exists a smooth map f such that $R_r \cong f^*E_r$, $g_{r+1} \cong f^*K_{r+1}$, $T \cong f^*A$, where $\cong$ stands for bundle equivalence. Moreover, if we denote by $P(H_1)$ the reduction of the structure group of P_1 to H_1, we have also $P(H_1) \cong f^*P_{H_1}$.

Let T^* be the cotangent bundle of M. We define a vector bundle morphism

$$\delta\colon S^{r+1}T^* \to T^* \otimes S^rT^* , \tag{2.4}$$

as the composition of bundle maps

$$S^{r+1}T^* \to \overset{r+1}{\otimes} T^* \to T^* \otimes (\overset{r}{\otimes} T^*)T^* \otimes S^rT^* ,$$

where S^rT^* stands for the r-th symmetric product of T^*. Then for any vector bundle F over M the map δ extends to a vector bundle morphism

$$\delta\colon \overset{j-1}{\wedge} T^* \otimes S^{r+1}T^* \otimes F \to \overset{j}{\wedge} T^* \otimes S^rT^* \otimes F \tag{2.5}$$

by sending $t \otimes s \otimes f$ into $(-1)^{j-1}t \otimes \delta(s) \otimes f$. Notice that a bundle map δ is well defined if in formula (2.5) we replace T^* by any vector bundle.

From the construction it follows that g_{r+1} is a subbundle of $S^{r+1}T^* \otimes T$. It can be shown [6] that $\delta(\overset{j-1}{\wedge} T^* \otimes g_{r+1}) \subset \overset{j}{\wedge} T^* \otimes g_r$, $r \geq 0$, and in fact that, for the k-th order involutive pseudogroup Γ, the sequence

$$0 \longrightarrow g_{k+n} \xrightarrow{\delta} T^* \otimes g_{k+n-1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \overset{n}{\wedge} T^* \otimes g_k \longrightarrow 0 \tag{2.6}$$

is exact.

Because we shall use the explicit structure of the vector bundles to enter Spencer's resolution of the sheaf Θ of germs of Γ-vector fields on M, we now present the construction in the form which is convenient for our purposes. Let A^* be the dual of A. Then there is a bundle map

$$\delta\colon \overset{j-1}{\wedge} A^* \otimes K_{r+1} \to \overset{j}{\wedge} A^* \otimes K_r , \qquad 1 \leq j \leq n, 0 \leq r , \tag{2.7}$$

which can also be defined formally as

$$(\delta\kappa)(a_1 \wedge \cdots \wedge a_j) = \sum_{s=1}^{j} (-1)^s \kappa(a_1 \wedge \cdots \wedge \widehat{a_s} \wedge \cdots \wedge a_j)a_s$$

for any $\kappa \in \overset{j-1}{\wedge} A^* \otimes K_{r+1}$ and $a_1, \cdots, a_j \in A$.

The dot operation is induced in an obvious way by restricting to the subbundle $\overset{j-1}{\wedge} A^* \times K_{r+1}$ the map

$$\overset{j-1}{\wedge} A^* \times S^{r+1}A^* \times A \to \overset{j}{\wedge} A^* \times S^r A^* \times A ,$$

which comes from the map

$$\underbrace{A^* \times \cdots \times A^*}_{j-1} \times \underbrace{A^* \times \cdots \times A^*}_{r+1} \times A \to \underbrace{A^* \times \cdots \times A^*}_{j} \times \underbrace{A^* \times \cdots \times A^*}_{r} \times A .$$

Proposition 2.2. *If Γ is a transitive elliptic involutive pseudogroup of order k, and $\mathscr{C}^i_{k-1} \to B_{H_1}$, $0 \leq 1 \leq n$, are the vector bundles defined by*

$$\mathscr{C}^i_{k-1} = (\overset{i}{\wedge} A^* \times E_k)\delta(\overset{i-1}{\wedge} A^* \times K_{k+1}) , \quad i \geq 1; \quad \mathscr{C}^0_{k-1} = E_k , \tag{2.8}$$

then there exists an exact sequence of sheaves

$$0 \longrightarrow \Theta \longrightarrow C^0_{k-1} \xrightarrow{D} C^1_{k-1} \xrightarrow{D} \cdots \xrightarrow{D} C^n_{k-1} \longrightarrow 0 \tag{2.9}$$

where

$$C^i_{k-1} = f^*\mathscr{C}^i_{k-1} ,$$

and the D's are first order differential operators.

Proof. For the construction of sequence (2.9) see [6]; the exactness of this sequence follows from the third fundamental theorem (see [7]).

3. Algebraic structure of C^i_{k-1}

The Lie bracket [,] on the vector fields on M defines a bilinear map $R_r \times R_r \to R_{r-1}$, $r \geq 0$, which we will denote by $[\![\ ,\ ]\!]$. It can be explicitely described as follows: Let σ_1, σ_2 be elements of $R_r | q$, $q \in M$. Then there are Γ-vector fields A_1 and A_2 on M such that $\sigma_i = j^r_q A_i$, $i = 1, 2$. Thus $[\![\sigma_1, \sigma_2]\!] = -j^{r-1}_q[A_1, A_2]$. There is an obvious extension of the operation to the R_r-valued differential forms. Let us denote by R^p_r the bundle $\overset{p}{\wedge} T^* \otimes R_r$. The extension of $[\![\ ,\ ]\!]$ is the operation (which we continue to denote by the same symbol) as a bilinear mapping $R^p_r \times R^q_r \to R^{p+q}_{r-1}$ such that for $\rho_i = \alpha_i \times \sigma_i$, $i = 1, 2$, $[\![\rho_1, \rho_2]\!] = \alpha_1 \wedge \alpha_2 \otimes [\![\sigma_1, \sigma_2]\!]$. Therefore $[\![\ ,\ ]\!]$ defines on the projective limit $R^*_\infty = \lim R^*_r$, $R^*_r = \bigoplus R^p_r$ a structure of graded Lie algebra.

The vector bundles C^i_{r-1}, $r \geq k$, from resolution (2.9) can be interpreted as bundles of derivations of degree i of the exterior algebra of differential forms on P_{r+1}. Each element $u \in C^i_{r-1}$ can be represented by a pair $u = (\sigma, \xi)$ with $\sigma \in R^i_{r-1}$ and $\xi = d\sigma - D\sigma_r$, where $\sigma_r \in R^i_r$ with projection $\pi(\sigma_r) = \sigma$, and D is the canonical first order operator from R^i_r to R^{i+1}_{r-1}, [9]. The anti-commutator $a \cdot b - b \cdot a$ gives a bracket operation on derivations of the exterior algebra on P_{r+1}, which induces a bracket [,] on $C^*_{r-1} = \bigoplus C^i_{r-1}$ and defines a structure of graded Lie algebra on C^*_{r-1}. There is a surjective bundle map $\sharp: R^i_r \to C^i_{r-1}$ and an operation $[\ ,\ ]: R^p_r \times R^q_r \to R^{p+q}_r$ such that for $\rho_1, \rho_2 \in R^*_r$, $[\sharp\rho_1, \sharp\rho_2] = \sharp[\rho_1, \rho_2]$, [8]. The operations $[\![\ ,\ ]\!]$ and $[\ ,\ ]$ give on R^*_∞ two graded Lie algebra structures, and are related in a simple way. Let us denote by π the projection $R^*_\infty \to R^*_0$, and, for any $\sigma \in R^p_\infty$ and $\tau \in R^q_\infty$, define $D_{\pi\sigma}\cdot\tau = D\tau \barwedge \pi\sigma + (-1)^p D(\tau \barwedge \pi\sigma)$. Then there is the formula

$$(3.1)\qquad \begin{aligned}[\sigma, \tau] &= D_{\pi\sigma}\cdot\tau - (-1)^{pq} D_{\pi\tau}\cdot\sigma + \sigma[\![\ ,\tau]\!] \\ &\quad + (-1)^{pq+q}\sigma \barwedge \pi(D\tau) - (-1)^q \tau \barwedge \pi(D\sigma)\ .\end{aligned}$$

Thus we obtain another Lie algebra structure on R^*_∞: $R^p_r \times R^q_r \to R^{p+q}_r$, which is denoted by the bracket $[\ ,\ ]$ and can be defined by the formula

$$(3.2)\qquad [\sigma, \tau] = D_{\pi\sigma}\cdot\tau - (-1)^p 2 D_{\pi\tau}\cdot\sigma + [\![\sigma, \tau]\!]\ , \qquad \sigma \in R^p_\infty,\ \tau \in R^q_\infty\ .$$

The following two propositions follow, by a direct computation, from the definitions.

Proposition 3.1. *For any $\sigma \in R^0_r$, $\tau \in R^q_r$ and any real-valued function f,*

$$(3.3)\qquad [\sigma, f\tau] = (df)\tau \barwedge \pi\sigma + f[\sigma, \tau]\ .$$

From (3.3) it follows in particular that if X is a Γ-vector field on M with a zero point p, then for any $\sigma \in R^q_r$ and any real valued function f we have $[j_r X, f\sigma](p) = f(p)[j_r X, \sigma]$. Therefore the resolution of the operator $L_X: R^q_r \to R^q_r$, given by

$$(3.4)\qquad L_X = [j_r X,\ ]\ ,$$

to the fibre of R^q_r over a zero point of X is an endomorphism of the fibre. Notice that the endomorphism L_X is well defined by (3.4) for $r \geq 0$, and can be described even more explicitly.

Proposition 3.2. *Let X be a Γ-vector field which is zero at a point p on M, and let $\sigma = j^r Y$ for some Γ-vector field Y. Then*

$$(3.5)\qquad (L_X\sigma)(p) = j^r_p[X, Y]\ .$$

Proof. Follows directly from the fact that $(L_X f\sigma)(p) = f(p)(L_X\sigma)(p)$ and from the definition of the "ordinary" bracket $[\ ,\ ]$ on vector fields.

Remarks. 1. All the Lie algebra structures defined here on R^*_r, $r \geq k$, are of course well defined already for $r \geq 0$.

2. The operator L_X given by (3.4) has a natural extension to $\overset{*}{\wedge} T^* \otimes R_r$, which is given as an operator

$$(3.6)\qquad \tilde{L}_X = L_X \otimes 1 + 1 \otimes L_X\ ,$$

where L_X stands for the Lie derivative along X. We shall use the notation L_X also for the extension $\tilde{L}_X$, unless there could arise some misunderstanding.

4. Singular forms

In this section we use the ideas of the paper [1] in order to relate the singular differential forms on the manifold M to the cohomology $H^*(M, R)$.

Definition 4.1. We call an r-pair (Θ, θ) of forms on M, $r \geq 0$, a pair consisting of a smooth $(r-1)$-form θ, defined on M except a nowhere dense set $e(\theta)$, and an extension Θ of $d\theta$ to $M - e(\Theta)$, $e(\Theta)$ being a subset of $e(\theta)$. For $r = 0$, putting $\theta \equiv 0$ we have $e(\theta) = \emptyset$. The sets $e(\Theta)$ and $e(\theta)$ lie on smooth locally finite polyhedra of dimensions $\leq n - r - 1$ and $n - r$ respectively.

A singular r-chain c on M with real coefficients is said to be admissible for the r-pair (Θ, θ), if the support $|c|$ of c has zero intersection with $e(\Theta)$, and the support $|\partial c|$ of the boundary ∂c of c does not intersect $e(\theta)$. Then we define the residue of (Θ, θ) with respect to an admissible r-chain c as the number

$$R[(\Theta, \theta), c] = \int_c \Theta - \int_{\partial c} \theta \,. \tag{4.1}$$

There can be defined an equivalence relation in the set of all r-pairs on M. The r-pairs (Θ_1, θ_1) and (Θ_2, θ_2) are equivalent if $R[(\Theta_1, \theta_1), c] = R[(\Theta_2, \theta_2), c]$ for each r-chain c which is admissible for both pairs. The set of equivalence classes $[\Theta, \theta]$ of the r-pairs (Θ, θ) form an R-module $C^r(M, R)$, $r \geq 0$. Define the exterior differential d to be the operation

$$d\colon C^r(M, R) \to C^{r+1}(M, R)$$

given by

$$d([\Theta, \theta]) = [0, \Theta] \,.$$

It is immediate that $d \cdot d = 0$. The kernel of the homomorphism is given by

$$Z^r(M, R) = \Big\{[\Theta, \theta] \,|\, \text{for any representation } (\Theta, \theta), \quad \int_{\partial c_{r+1}} \Theta = 0 \text{ if } |\partial c_{r+1}| \cap e(\Theta) = \emptyset\Big\} \,.$$

If we denote by $\mathscr{H}(M, R)$ the cohomology algebra of r-pairs, and $H(M, R)$ the real cohomology algebra, one prove

Proposition 4.1 [1]. *Let M be a manifold. Then there is a canonical isomorphism*

$$\mathscr{H}(M, R) = H(M, R) \,. \tag{4.2}$$

Let (Θ, θ) be an r-pair, which represents a singular r-cycle $[\Theta, \theta] \in Z^r(M, R)$ and therefore the cohomology class γ of $H^r(M, R)$ via the isomorphism (4.2). We shall talk about the residue of γ defined by

$$\text{Res}\,\gamma = R[(\Theta, \theta), c] \tag{4.3}$$

for an admissible r-chain c. We also refer to Res γ as to the characteristic number of γ.

5. Characteristic numbers associated with a vector field

In this section we define the important representatives for certain cohomology classes in the top cohomology $H^n(M, R)$. Suppose that we are given a Γ-vector field X on M with the set S of isolated zeros of order 0. It turns out that such a vector field defines in a natural way a characteristic class $\alpha(X) \in H^n(M, R)$, and that the characteristic number of $\alpha(X)$ is computable in terms of the eigenvalues of the Lie derivative L_X on $T(M)$, restricted to S.

Because over $\overline{M} = M - S$ we have a nonvanishing vector field X, the anihilator $X^\#$ of X is a well defined subbundle of 1-forms $A^1(\overline{M})$ over $\overline{M}$. Then there is an exact sequence

$$0 \to X^\# \to A^1(\overline{M}) \to X^* \to 0\,. \tag{5.1}$$

Now we show that there is a representative π of the class X^* in $A^1(\overline{M})$ such that the following theorem holds.

Theorem 5.1. *Let X be a vector field (not necessarily a Γ-vector field) on an oriented manifold M with the set S of N isolated zero points, where X vanishes up to order zero and the endomorphisms of $T_p(M)$, $p \in S$, given by the Lie derivative L_X have eigenvalues $\lambda_1(p), \cdots, \lambda_n(p)$. Then there exists a 1-form π on M, with singularities at the points of S, which is a representative for the class X^* in $A^1(\overline{M})$ such that via the isomorphism* (4.2)

$$[\underbrace{d\pi \wedge \cdots \wedge d\pi}_{l}, \pi \wedge \underbrace{d\pi \wedge \cdots \wedge d\pi}_{l-1}]$$

represents the cohomology class $\alpha(X) \in H^n(M, R)$, and

$$\text{Res}\,\alpha(X) = \sum_{p \in S} (\lambda_1(p) \cdots \lambda_n(p))^{-1/2} \cdot (2^{l-1}(l-1)!\,Nk(n))\,, \tag{5.2}$$

where $k(n)$ is a constant which does not depend on X.

Proof. There are a neighborhood U_p of a point $p \in S$ and a coordinate system $(x_1, \cdots, x_n)$ in U_p such that the modulo terms which vanish at least up to the second order are

$$X\,|\,U_p \sim \sum_{i=1}^{n} \lambda_i x_i \partial/\partial x_i\ , \quad \lambda_i = \lambda_i(p)\ , \quad 1 \le i \le n\ .$$

Then there are functions Λ_i such that

$$X\,|\,U_p = \sum_{i=1}^{n} x_i \Lambda_i \partial/\partial x_i\ .$$

Assume that the neighborhoods U_p, $p \in S$, are mutually disjoint. Let there be a riemannian metric on M such that the inner product (,) on $T(M)\,|\,U_p$ is given by the formulas

$$(5.3)\qquad (\partial/\partial x_i, \partial/\partial x_j) = \delta_{ij}\ , \qquad 1 \le i, j \le n\ .$$

In the neighborhood U_p, define the new inner product [,] by

$$(5.4)\qquad [\partial/\partial x_i, \partial/\partial x_j] = (\Lambda_i \Lambda_j)^{-1}(\partial/\partial x_i, \partial/\partial x_j)$$

and the endomorphism $A_p\colon T(M)\,|\,U_p \to T(M)\,|\,U_p$ by

$$(5.5)\qquad \begin{aligned} A_p \partial/\partial x_{2\alpha-1} &= (\Lambda_{2\alpha-1}\lambda_{2\alpha})^{-1/2} \Lambda_{2\alpha-1}^{-1} \Lambda_{2\alpha}^{+2} \partial/\partial x_{2\alpha}\ , \\ A_p \partial/\partial x_{2\alpha} &= -(\lambda_{2\alpha-1}\lambda_{2\alpha})^{-1/2} \Lambda_{2\alpha-1}^{+2} \Lambda_{2\alpha}^{-1} \partial/\partial x_{2\alpha-1}\ , \end{aligned}$$

$1 \le \alpha \le l$. Now let us define a 1-form π_p over U_p:

$$(5.6)\qquad \pi_p = [A_p(X),\ \ ]/[X, X]\ .$$

There exist a smooth function ϕ_p and a closed neighborhood $V_p \subset U_p$ of the point p such that $\phi_p \equiv 1$ on V_p and $\phi_p \equiv 0$ on $M - U_p$. Then define

$$(5.7)\qquad \pi = \sum_{p \in S} \phi_p \pi_p\ .$$

This 1-form is obviously singular at all points of S. But it turns out that the n-form $d(\pi \wedge d\pi \wedge \cdots \wedge d\pi)$ can be smoothly extended across the singular points, as follows from the observation: On U_p

$$\begin{aligned} A_p(X) &= \sum_{i=1}^{n} x_i \Lambda_i A_p \partial/\partial x_i \\ &= \sum_{\alpha=1}^{l} \{x_{2\alpha-1}(\lambda_{2\alpha-1}\lambda_{2\alpha})^{-1/2} \Lambda_{2\alpha}^{+2} \partial/\partial x_{2\alpha} \\ &\qquad\qquad - x_{2\alpha}(\lambda_{2\alpha-1}\lambda_{2\alpha})^{-1/2} \Lambda_{2\alpha-1}^{+2} \partial/\partial x_{2\alpha-1}\}\ , \end{aligned}$$

$$[A_p(X),\ \ ] = \sum_{\alpha=0}^{l} \{x_{2\alpha-1}(\lambda_{2\alpha-1}\lambda_{2\alpha})^{-1/2} dx_{2\alpha} - x_{2\alpha}(\lambda_{2\alpha-1}\lambda_{2\alpha})^{-1/2} dx_{2\alpha-1}\}\ ,$$

and due to $[X, X] = \sum_{i=1}^{n} (x_i)^2$ we get

$$\left(\sum_{i=1}^{n}(x_i)^2\right)\pi_p = [A_p(X),\ \ ]\,,$$

and finally the formula

$$\left(\sum_{i=1}^{n} x_i^2\right)^l \pi_p \wedge d\pi_p \wedge \cdots \wedge d\pi_p$$
$$= \frac{2^{l-1}(l-1)!}{(\lambda_1(p)\cdots\lambda_n(p))^{1/2}} \sum_{i=1}^{n}(-1)^{i+1}x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\,.$$

Now it is just a matter of direct computation to check that actually $d(\pi_p \wedge d\pi_p \wedge \cdots \wedge d\pi_p) = 0$.

Let us denote by $B_\varepsilon(p)$ the ball with the center p and radius ε, $B_\varepsilon(p) \subset U_p$, and let its boundary $\partial B_\varepsilon(p)$ be oriented consistently with the orientation of M. If we denote by $k(n)$ the area of the unit $(n-1)$-sphere in M with center in $p \in S$, then we get from [1] that

$$(5.8)\quad \int_{\partial B_\varepsilon(p)} \sum_{i=1}^{n}(-1)^{i+1}x_i\left(\sum_{j=1}^{n}x_j^2\right)^{-l} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n = +k(n)$$

for all small ε. Therefore without any restriction we can assume that $B_\varepsilon(p) \subset V_p$. If we choose such a ball for all $p \in S$, then $B_\varepsilon = \bigcup_{p\in S} B_\varepsilon(p)$ is a singular n-chain on M with the orientation given by the orientation of M, and the residue of the n-pair $(d\pi \wedge d\pi \wedge \cdots \wedge d\pi, \pi \wedge d\pi \wedge \cdots \wedge d\pi)$ with respect to B_ε can be computed. We see that

$$\int_{B_\varepsilon} d\pi \wedge \cdots \wedge d\pi - \int_{\partial B_\varepsilon} \pi \wedge d\pi \wedge \cdots \wedge d\pi$$
$$= -\sum_{p\in S} \int_{\partial B_\varepsilon(p)} \pi_p \wedge d\pi_p \wedge \cdots \wedge d\pi_p$$
$$= \sum_{p\in S} \frac{2^{l-1}(l-1)!}{(\lambda_1\cdots\lambda_n)^{1/2}} \int_{\partial B_\varepsilon(p)} \sum_{i=1}^{n}(-1)^{i+1}x_i\left(\sum_{j=1}^{n}x_j^2\right)^{-l}$$
$$\cdot dx_1 \wedge \cdots \wedge dx_i \wedge \cdots \wedge dx_n$$
$$= 2^{l-1}(l-1)!\, Nk(n) \sum_{p\in S}(\lambda_1(p)\cdots\lambda_n(p))^{-1/2}\,.$$

6. Topological index

In this section we give an explicit formula for the Euler characteristic $\chi(M, \Theta)$ which can be looked at as a topological index of certain elliptic differential operator associated to the given elliptic pseudogroup Γ on a compact oriented manifold M. To be more specific let us put a riemannian metric along the fibres of the vector bundles $T = T(M)$ and R_k corresponding to the chosen reduction of the structure group G_1 to H_1. This induces a metric along the fibres of C^i_{k-1} and the global product by its integral. Let D^* be the adjoint operator to D with respect to this global product. Then there is an operator

$$(6.1) \qquad D + D^* : \sum_{s=0}^{l-1} C^{2s}_{k-1} \to \sum_{s=0}^{l-1} C^{2s+1}_{k-1}$$

on M. This operator can be extended to the complexification

$$(6.2) \qquad \begin{aligned} E &= \sum_{s=0}^{l-1} C^{2s}_{k=1} \otimes C = f^*\left(\sum_{s=0}^{l-1} \mathscr{C}^{2s}_{k-1} \otimes C\right) = f^*(E) , \\ F &= \sum_{s=0}^{l-1} C^{2s+1}_{k-1} \otimes C = f^*\left(\sum_{s=0}^{l-1} \mathscr{C}^{2s+1}_{k-1} \otimes C\right) = f^*(F) \end{aligned}$$

as the operator $(D + D^*) \otimes \mathrm{id}$. We shall write briefly

$$(6.3) \qquad D + D^* : E \to F ,$$

and denote by its symbol $\sigma(D + D^*)$. Then we have the Thom isomorphism

$$(6.4) \qquad \phi_* : H^i(M, Q) \to H^{i+n}(B(M), S(M); Q) .$$

Because the pseudogroup is elliptic, it can be shown that $D + D^*$ is an elliptic operator, and therefore the symbol $\sigma(D + D^*)$ is an injective map from π^*E to π^*F. It has to be pointed out that the homotopy class of the isomorphism $\sigma(D + D^*)$ does not depend on the choice of the riemannian metric. Then there is a unique element in the relative K-group

$$d(\pi^*E, \pi^*F, \sigma(D + D^*)) \in K(B(M), \sigma(M)) .$$

If we denote by $\mathrm{td}\, T$ the Todd class of $T = T \otimes C$, and by $\mathrm{ch}: K(X, Y) \to H^*(X, Y; Q)$ the Chern character, then the topological index $i_t(D + D^*)$ of the operator $D + D^*$ is defined as the value of the cohomology class

$$\phi_*^{-1} \mathrm{ch}\, (d(\pi^*E, \pi^*F, \sigma(D + D^*)) \cdot \mathrm{td}\, T = \phi_*^{-1} \mathrm{ch}\, (D + D^*) \cdot \mathrm{td}\, T$$

on the fundamental cycle $[M]$ of the oriented compact manifold M. In order to get an explicit formula we simplify the expression for $\mathrm{ch}\,(D + D^*)$. The first step is done in

Proposition 6.1 [9]. *Let E and F be the vector bundles over B_{H_1} defined by* (6.2), *and let $e(A)$ be the Euler class of the vector bundle $A \to B_{H_1}$. Then*

$$(6.5) \qquad \begin{aligned} \phi_*^{-1} \mathrm{ch}\, (D + D^*) &= \phi_*^{-1} \mathrm{ch}\, (d(\pi^*E, \pi^*F, \sigma(D + D^*))) \\ &= f^{**} \frac{\mathrm{ch}\, E - \mathrm{ch}\, F}{e(A)} . \end{aligned}$$

We need a little more than just the universality of the symbol $\sigma(D + D^*)$

required in the previous proposition in order to get more explicit formula for the right hand side of (6.5). Let us look at the symbol $\sigma(D)$ itself. Denote by $V = V_0$ the real H_1-module, and by B^i, L_l the typical fibres of $\mathscr{C}^i_{k-l} \otimes C$ and $K_{k+l} \otimes C$ respectively. If we take A^*, $\mathscr{C}^i_{k-l} \otimes C$ instead of T^* and F in formula (2.5), restrict to a fibre and take $j = 1$ we get the H_1-morphism

$$\delta \colon S^{r+1}V^* \otimes B^i \to S^rV^* \otimes V^* \otimes B^i \,, \tag{6.6}$$

$r \geq 0$, $0 \leq i \leq n$. This map, composed with the symbol

$$\sigma(D) \colon V^* \otimes B^i \to B^{i+1} \,, \qquad 0 \leq i \leq n \,, \tag{6.7}$$

gives

$$\tau^r_i = (1 \otimes \sigma(D)) \cdot \delta \colon S^{r+1}V^* \otimes B^i \to S^rV^* \otimes B^{i+1} \,. \tag{6.8}$$

The injection

$$i^l \colon L_l \to S^lV^* \otimes B^0 \tag{6.9}$$

($B^0 = V_k \otimes C$), $r \geq 1$, is obviously also an H_1-map. Notice that $i^1 = \delta$. From the general theory [6] it follows that sequence (2.9) is formally exact, so that for any $l \geq 1$ we have the sequence

$$\begin{aligned} 0 \longrightarrow R_{k+l} \longrightarrow J_l(C^0_{k-1}) \xrightarrow{p_{l-1}(D^0)} J_{l-1}(C^1_{k-1}) \xrightarrow{p_{l-2}(D^1)} \\ \cdots \xrightarrow{p_{l-r}(D^{r-1})} J_{l-r}(C^r_{k-1}) \end{aligned} \tag{6.10}$$

which is exact at R_{k+l} for $l \geq 1$ and at $J_{l-i}(C^i_{k-1})$ for $l \geq i+1$, $0 \leq i \leq r-1$. The bundle maps $p_{l-i}(D^{i-1})$ are defined by the commutative diagram

$$\begin{array}{ccc} J_{l-i+1}(C^{i-1}_{k-1}) & \xrightarrow{p_{l-i}(D^{i-1})} & J_{l-i}(C^i_{k-1}) \\ \uparrow{\scriptstyle j^{l-i+1}} & & \uparrow{\scriptstyle j^{l-i}} \\ C^{i-1}_{k-1} & \xrightarrow{D^{i-1}} & C^i_{k-1} \end{array} \tag{6.11}$$

where we write D^i instead D to make clear on which space the operator acts. The commutative diagram

$$\begin{array}{ccccccccc} 0 \to & S^{l+1}T^* \otimes C^i_{k-1} & \to & J_{l+1}(C^i_{k-1}) & \to & J_l(C^i_{k-1}) & \to 0 \\ & \downarrow{\scriptstyle \sigma_l(D^i)} & & \downarrow{\scriptstyle p_l(D^i)} & & \downarrow{\scriptstyle p_{l-1}(D^i)} & \\ 0 \to & S^lT^* \otimes C^{i+1}_{k-1} & \to & J_l(C^{i+1}_{k-1}) & \to & J_{l-1}(C^{i+1}_{k-1}) & \to 0 \end{array} \tag{6.12}$$

defines a bundle map

(6.13) $$\sigma_l(D^i)\colon S^{l+1}T^*\otimes C^i_{k-1}\to S^lT^*\otimes C^{i+1}_{k-1}$$

for $0\le i\le n-1$, $0\le l$, $\sigma_0(D^i)=\sigma(D)$, such that $\sigma_l(D^i)=(1\otimes\sigma(D^i))\cdot\delta$. Therefore by restricting to a fibre we get

(6.14) $$\sigma_l(D^i)\,|\,0=\tau^l_i\ ,$$

and from (6.10) and (6.12) we have the exact sequence of vector bundles

(6.15) $$0\longrightarrow g_{k+l}\xrightarrow{i^l}S^lT^*\otimes C^0_{k-1}\xrightarrow{\sigma_{l-1}(D^0)}S^{l-1}T^*\otimes C^1_{k-1}\xrightarrow{\sigma_{l-2}(D^1)}\cdots\longrightarrow\sigma_{l-r}(D^{r-1})S^{l-r}T^*\otimes C^r_{k-1}\ ,$$

where i^l is an inclusion for $l>1$ and $i^1=\delta$. Now let us consider the complexification of all the vector bundles in (6.15) with the obvious extension of the operators so that the new sequence remains exact, i.e.,

(6.16) $$0\longrightarrow g_{k+l}\otimes C\xrightarrow{i^l}S^lT^*\otimes C^0_{k-1}\otimes C\xrightarrow{\sigma_{l-1}(D^0)}\cdots\ .$$

If we restrict this complex extension to the origin $0\in M$ and use the notation (6.6), we get an exact sequence of vector spaces, with all the maps being H_1-maps,

(6.17) $$0\longrightarrow L_l\xrightarrow{i^l}S^lV^*\otimes B^0\xrightarrow{\tau_0^{l-2}}S^{l-1}V^*\otimes B^1\xrightarrow{\tau_1^{l-2}}\cdots\xrightarrow{\tau_{r-1}^{l-r}}S^{l-r}V^*\otimes B^r\ .$$

All the bundles in sequence (6.15) are associated to the principal H_1-bundle $P_1(H_1)\to M$. The complexification of these vector bundles is associated to the principal bundle $P_1^C=P_1^C(H_1)\to M$, which is the h-extension of $P_1(H_1)$ and is given as follows: Because H_1 is a subgroup of $SO(n)$ let us consider the natural homomorphism

$$h\colon SO(n)\to U(n)\ .$$

This gives a map

$$h\colon H^1(M,H_1)\to H^1(M,h(H_1))\ ,$$

and $P_1{}^C$ is then given by this homomorphism from $P_1(H_1)$ up to the equivalence. Let us denote the h-extension of P_{H_1} by $P^C_{H_1}\to B_{H_1}$. We can certainly find the representatives in the equivalence classes so that the classifying map f for the real principal bundle P_1 gives

$$f^{-1}P^C_{H_1} = P_1{}^C , \tag{6.18}$$

where $P^C_{H_1}$ is the h-extension of P_{H_1}.

Now the complexification of (6.15) is the exact sequence

$$\begin{aligned} 0 \longrightarrow P_1{}^C x_{H_1} L_l \xrightarrow{i^l} P_1{}^C x_{H_1} S^l V^* \otimes B^0 \xrightarrow{\tau_0^{l-1}} \cdots \\ \xrightarrow{\tau_{r-1}^{l-r}} P_1{}^C x_{H_1} S^{l-r} V^* \otimes B^r , \end{aligned} \tag{6.19}$$

and it is the pull beck, by f, of the exact sequence of vector bundles over B_{H_1},

$$\begin{aligned} 0 \longrightarrow P^C_{H_1} x_{H_1} L_l \xrightarrow{i^l} P^C_{H_1} x_{H_1} S^l V^* \otimes B^0 \xrightarrow{\tau_0^{l-1}} \cdots \\ \xrightarrow{\tau_{r-1}^{l-r}} P^C_{H_1} x_{H_1} S^{l-r} V^* \otimes B^r . \end{aligned} \tag{6.20}$$

Notice that $P^C_{H_1} x_{H_1} S^l V^* \otimes B^r = S^l A^* \otimes (\mathscr{C}^r_{k-1} \otimes C)$, and H_1 acts on $S^l V^* \otimes B^r$ by the representation $H_1 \otimes h(H_1)$. If we denote briefly $\mathscr{C}^r_{k-1} \otimes C$ by $\mathscr{C}^r$, τ^{l-i}_{i-1} simply by τ^{l-i} and $P^C_{H_1} x_{H_1} L_l$ by $\mathscr{L}_l$, we can summerize our results in the following form:

Proposition 6.2.

$$\begin{aligned} 0 \longrightarrow \mathscr{L}_l \xrightarrow{i^l} S^l A^* \otimes \mathscr{C}^0 \xrightarrow{\tau^{l-1}} S^{l-1} A^* \otimes \mathscr{C}^1 \xrightarrow{\tau^{l-2}} \cdots \\ \xrightarrow{\tau^{l-r}} S^{l-r} A^* \otimes \mathscr{C}^r \end{aligned} \tag{6.21}$$

is an exact sequence of vector bundles over B_{H_1}, for $l \geq 1, 0 \leq r \leq l$. The pull beck of this sequence, by the classifying map f, is the complexification (6.19) *of* (6.15).

It is important to remember that while $\mathscr{L}_l$'s and $\mathscr{C}^i$'s are complex vector bundles over B_{H_1}, the vector bundles $S^l A^*$ are real and $\otimes$ stands for $\otimes_R$.

Lemma 6.1. *There are polynomials P_j, $0 \leq j \leq n$, $P_0 = +1$, in the Chern characters of the vector bundles $A^*, \cdots, \bigwedge^j A^*$ such that*

$$\operatorname{ch} E - \operatorname{ch} F = \sum_{j=1}^{n} \operatorname{ch} \mathscr{L}_j \cdot P_{n-j} + \operatorname{ch} \mathscr{C}^0 \cdot (P_n - 1) . \tag{6.22}$$

Proof. From (6.21) we get, for $l = r$, an exact sequence with the last map τ^0: $A^* \otimes \mathscr{C}^{l-1} \to \mathscr{C}^l$ surjective and with all maps bundle morphisms. Then we can compute $\operatorname{ch} \mathscr{C}^l$ in terms of $\operatorname{ch} \mathscr{L}_l$ and $\operatorname{ch}(S^{l-i} A^* \otimes \mathscr{C}^i)$, $0 \leq i \leq l$, from the formula

$$\operatorname{ch} \mathscr{L}_l - \operatorname{ch}(S^l A^* \otimes \mathscr{C}^0) + \operatorname{ch}(S^{l-1} A^* \otimes \mathscr{C}^1) - \cdots + (-1)^{l-1} \operatorname{ch} \mathscr{C}^l = 0 .$$

Furthermore, we have

$$\begin{aligned}\mathrm{ch}\,(S^{l-i}A^*\otimes C)\cdot\mathrm{ch}\,\mathscr{C}^i &= \mathrm{ch}\,((S^{l-i}A^*\otimes C)\otimes\mathscr{C}^i)\\ &= \mathrm{ch}\,((S^{l-i}A^*\oplus S^{l-i}A^*)\otimes\mathscr{C}^i) = 2\,\mathrm{ch}\,(S^{l-1}A^*\otimes\mathscr{C}^i)\,,\end{aligned}$$

observing that $S^{l-i}A^*\otimes C$ is isomorphic to $S^{l-i}A^*\oplus S^{l-i}A^*$. This gives

$$\begin{aligned}\mathrm{ch}\,\mathscr{C}^l = \frac{(-1)^l}{2}\{2\,\mathrm{ch}\,\mathscr{L}_l - \mathrm{ch}\,(S^lA^*\otimes C)\cdot\mathrm{ch}\,\mathscr{C}^0 - \cdots\\ + (-1)^l\,\mathrm{ch}\,(A^*\otimes C)\cdot\mathrm{ch}\,\mathscr{C}^{l-1}\}\,.\end{aligned}\tag{6.23}$$

From the defining equation (2.5) of δ follows the exactness of the sequence

$$\begin{aligned}0\to S^lA^*\otimes C\to S^{l-1}A^*\otimes C\to\cdots\\ \to A^*\otimes(\textstyle\bigwedge^{l-1}A^*\otimes C)\to\textstyle\bigwedge^{l}A^*\otimes C\to 0\,.\end{aligned}\tag{6.24}$$

Moreover, an inductive procedure gives $\mathrm{ch}\,(S^{l-i}A^*\otimes C)$ in terms of $\mathrm{ch}\,(\bigwedge^j A^*\otimes C)$, $l-i\geq j$. This together with (6.23) and the definition of E, F gives the formula (6.22).

Remark. A direct computation shows that

$$\begin{aligned}P_0 &= +1,\ P_1 = +1-\tfrac{1}{2}\,\mathrm{ch}\,(A^*\otimes C),\ \cdots,\\ P_4 &= +1-\tfrac{1}{2}\,\mathrm{ch}\,(A^*\otimes C)+\tfrac{1}{2}\,\mathrm{ch}\,(\textstyle\bigwedge^2 A^*\otimes C)-\tfrac{1}{4}\,\mathrm{ch}\,(\textstyle\bigwedge^3 A^*\otimes C)\\ &\quad+\tfrac{1}{8}\,\mathrm{ch}\,(\textstyle\bigwedge^4 A^*\otimes C)+\tfrac{5}{8}\,\mathrm{ch}\,(\textstyle\bigwedge^3 A^*\otimes C)\cdot\mathrm{ch}\,(A^*\otimes C)\\ &\quad-\tfrac{1}{2}\,\mathrm{ch}\,(\textstyle\bigwedge^2 A^*\otimes C)\cdot\{\mathrm{ch}\,(A^*\otimes C)\}^2+\tfrac{1}{2}\left\{\mathrm{ch}\,(\textstyle\bigwedge^2 A^*\otimes C)\right\}^2,\ \cdots\,.\end{aligned}$$

7. Fixed point formula for a nondegenerate Γ-vector field X

Suppose that X is a Γ-vector field with isolated zeros. Furthermore assume that Γ has the degree $k=1$, and that the Lie derivative $\mathscr{L}_X$ in the direction of X, as an operator on the tensor bundles over M, restricted to the zero point of X is a nonsingular endomorphism of the fibre. In that case we say that X is nondegenerate, and only nondegenerate Γ-vector field X with a set S of isolated zeros will be considered. The vector field X generates a 1-parameter group $\{f_t\}$ of transformations $f_t=\exp tX$. The following two propositions show that all the assumptions on the transformations $f_t\colon M\to M$ are satisfied for the Atiyah-Bott fixed point theorem to be applicable to this situation. From Proposition 2.2 follows that the vector bundles C^i_{k-1} in resolution (2.10) can be given by

$$C_0^i = (\overset{i}{\wedge} T^* \otimes R_1)\delta(\overset{i-1}{\wedge} T^* \otimes g_2) . \tag{7.1}$$

The differential of the transformation f_t maps Γ vector fields into Γ vector fields. Hence there is an induced bundle map of R_r, $r \geq k$, into itself, covering the transformation f_t. Because g_{r+1} is the kernel of the projection $R_{r+1} \to R_r$, the induced transformation on R_{r+1} maps g_{r+1} into itself. Furthermore from the definition (2.5) of δ it follows that the induced transformation commutes with δ. Therefore f_t has a natural lifting

$$\phi_t^i : f_t^{-1} C_{k-1}^i \to C_{k-1}^i , \qquad 0 \leq i \leq n . \tag{7.2}$$

If $\Gamma_{f_t} : C_{k-1}^i \to f_t^{-1} C_{k-1}^i$ is the natural transformation of sections of C_{k-1}^i into the sections of the induced bundle $f_t^{-1} C_{k-1}^i$ we define an endomorphism

$$T_t^i = T(f_t, \phi_t^i) = \phi_t^i \cdot \Gamma_{f_t} : C_{k-1}^i \to C_{k-1}^i . \tag{7.3}$$

We denote by the same symbol the mapping induced on sheaves of germs of smooth sections.

Proposition 7.1. *The first order operator* $D: C_{k-1}^i \to C_{k-1}^{i+1}$ *commutes with* T_t^i, *in other words,*

$$T_t^{i+1} D = D T_t^i . \tag{7.4}$$

Proof. First of all observe that the map T_t^i can be lifted, in an obvious way, to a map

$$T_t^i : \overset{i}{\wedge} T^* \otimes R_k \to \overset{i}{\wedge} T^* \otimes R_k$$

such that $T_t^i(\delta(\overset{i-1}{\wedge} T^* \otimes g_{k+1})) \subset \overset{i}{\wedge} T^* \otimes g_k$, and that it commutes with the projection $p: \overset{i}{\wedge} T^* \otimes R_k \to C_{k-1}^i$. Since T_t^i is a composition of two maps, namely, the natural transformation and the lifting of f_t, the commutativity of the natural transformation with D is obvious. The lifting of f_t commutes with D because D is essentially the sum of two operators, namely, ordinary differential and δ, which commute with the lifting. Proof of this point is done by direct computation, using the explicit description of D in [8]. q.e.d.

From now on let us restrict ourselves to the case $k = 1$. We shall use much finer description of the lifting $\phi_t^i : f_t^{-1} C_0^i \to C_0^i$, $0 \leq i \leq n$. For this reason we look at the structure of C_0^i more closely. The basic observation is that once a splitting λ of the exact sequence

$$0 \to g_1 \to R_1 \to R_0 \to 0 \tag{7.5}$$

is chosen, there is an isomorphism [9]

$$\iota : C_0^i \to (\overset{i}{\wedge} T^* \otimes R_0) \oplus \delta(\overset{i}{\wedge} T^* \otimes g_1) . \tag{7.6}$$

Proposition 7.2. *A splitting λ of the exact sequence* (7.5) *can be chosen in such a way that, when restricted to the fibres over the zero points of X, it commutes with the transformations induced by the* 1*-parameter group* $\{f_t\}$.

Proof. Let $(x^1, \cdots, x^n)$ be the coordinates of a point x in a neighborhood U of a zero point p of X, and $Y_1, \cdots, Y_n$ be nonvanishing Γ-vector fields on U. Then define a splitting $\lambda_U \colon R_0 | U \to R_1 | U$ by

$$\lambda_U(Y_i) = j^1 Y_i \,. \tag{7.7}$$

By a direct computation it is easy to see that λ_U commutes with f_t over the whole U.

Let $U = \{U_i\}$ be a covering of M such that for each zero point p of X there is a neighborhood $W(p) \subset U_{i_p}$, U_{i_p} being the open set of the covering U, $W(p) \cap U_i = \emptyset$, $i \neq i_p$; $W(p) \cap W(q) = \emptyset$ for $p \neq q$. This can be done because S is the set of isolated points. Consider now a partition of unity $\{\phi_i\}$ associated to such U. If λ_i is the splitting over U_i, defined by (7.7), then $\lambda = \Sigma \phi_i \lambda_i$ splits (7.5) and commutes with f_t on some $V(p) \subset W(p)$. Therefore the induced transformation $R_1(p)$ on $R_0(p)$ commutes with λ also.

Corollary. *There exists an isomorphism* (7.6) *commuting with the natural liftings of f_t over the set S of zero points of X.*

Let us denote by

$$(T_t^i)^* \colon H^i(M, \Theta) \to H^i(M, \Theta) \tag{7.8}$$

the endomorphism induced by T_t^i and by the Jacobian $J_p(f_t)$ of f_t. The Atiyah-Bott fixed point theorem gives the formula

$$\sum_{p \in S} \sum_{i=0}^{n} (-1)^i \operatorname{trace} (T_t^i)^* = \sum_{p \in S} \sum_{i=0}^{n} (-1)^i \frac{\operatorname{trace} \phi_t^i(p)}{|\det (1 - J_p(f_t))|} \,. \tag{7.9}$$

Similarly as in the previous section we can give a more explicit form to the right hand site by considering the special situation arising from the given Γ-structure on M.

The lifting ϕ_t^i (7.2) is an operator on $(\bigwedge^i T^* \otimes R_0) \oplus \delta(\bigwedge^i T^* \otimes g_1)$ up to the isomorphism (7.6). Let us denote by π_1, π_2 the projections of this vector bundle onto the components, and define the operators (over $V(p)$—see the proof of Proposition 7.2)

$$\begin{aligned} \alpha_t^i &= \iota \phi_t^i f_t^{-1} \iota^{-1} \pi_1 \colon \bigwedge^i T^* \otimes R_0 \Big| V(p) \to \bigwedge^i T^* \otimes R_0 \Big| V(p) \,, \\ \beta_t^i &= \iota \phi_t^i f_t^{-1} \iota^{-1} \pi_2 \colon \delta(\bigwedge^i T^* \otimes g_1) \Big| V(p) \to \delta(\bigwedge^i T^* \otimes g_1) \Big| V(p) \,. \end{aligned} \tag{7.10}$$

Then

$$\text{trace}\,\phi_t^i(p) = \text{trace}\,\alpha_t^i(p) + \text{trace}\,\beta_t^i(p)\ . \tag{7.11}$$

Trace $\alpha_t^i(p)$ is easy to compute because

$$\alpha_t^i(p) = \bigwedge^i f_t^*(p) \otimes f_t(p)_*$$

so that

$$\text{trace}\,\alpha_t^i(p) = \text{trace} \bigwedge^i f_t^*(p) \cdot \text{trace}\, f_t(p)^*\ . \tag{7.12}$$

From the exact sequence of vector bundles (2.6)

$$0 \longrightarrow g_{i+1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \bigwedge^{i-1} T^* \otimes g_2 \xrightarrow{\delta} \bigwedge^i T^* \otimes g_1$$
$$\xrightarrow{\delta} \delta(\bigwedge^i T^* \otimes g_1) \longrightarrow 0$$

by inductive procedure it follows that

$$\text{trace}\,\beta_t^i(p) = \sum_{k=0}^i (-1)^k\, \text{trace} \bigwedge^{i-k} f_T^* \cdot \text{trace}\, f_t^{k+1}\ , \tag{7.13}$$

where $f_t^{k+1}\colon g_{k+1} \to g_{k+1}$ is induced by f_t, $f_t^0(p) = f_t(p)_*$. This shows that

$$\sum_{i=0}^n (-1)^i\, \text{trace}\,\phi_t^i(p) = \sum_{i=0}^n (-1)^i \Big\{ \text{trace} \bigwedge^i f_t^*(p) \cdot \text{trace}\, f_t^0(p)$$
$$+ \sum_{k=0}^i (-1)^k\, \text{trace} \bigwedge^{i-k} f_t^*(p) \cdot \text{trace}\, f_t^{k+1} \Big\}\ .$$

The Lie derivative L_X in the direction X induces on the fibre $T_p(M)$, $p \in S$, nonsingular endomorphism $L_p = L_X | T_p(M)$ given in (3.4). Then a direct computation shows that $f_t{}^0(p) = \exp tL_p$, and $f_t{}^* = \exp tL_p{}^*$. Denote by L_p^{k+1} the transformation induced by L_X on the fibre $g_{k+1}(p)$ defined by (3.4). Then $f_t^{k+1}(p) = \exp tL_p^{k+1}$. Assume furthermore that L_p has, for each $p \in S$, n nonvanishing eigenvalues $\lambda_1(p), \cdots, \lambda_n(p)$. If there is no danger of confusion we shall write simply $\lambda_1, \cdots, \lambda_n$. In a suitable local coordinate system around p the endomorphism L_p is represented by a matrix with the only nonzero elements $\lambda_1, \cdots, \lambda_n$ in the diagonal. These observations make it possible to give the Lefschetz number (7.9) in a more explicit form

Lemma 7.1. *The Euler characteristic*

$$\chi(M, \Theta) = \sum_{p \in S} \varepsilon_p \cdot \frac{1}{\det(1 - e^{tL_p})} \sum_{i=0}^n (-1)^i \Big\{ \text{tr} \bigwedge^i e^{tL_p} \cdot \text{tr}\, e^{tL_p}$$
$$+ \sum_{k=0}^i (-1)^k\, \text{tr} \bigwedge^{i-k} e^{tL_p^*} \cdot \text{tr}\, e^{tL_p^{k+1}} \Big\} \Big| (\textit{the constant term}) \tag{7.14}$$

where $\varepsilon_p = $ *sign* $\det(1 - e^{\iota L_p})$.

It turns out that we can get formally similar expression for the topological index $i_t(D + D^*)$. Notice, first of all, that the isomorphism ι in (7.6) extends tothe complexification of the bundles involved. Identify $C_0^i \otimes C$ with the ι-isomorphic bundle $(\overset{i}{\wedge} T^* \otimes R_0) \otimes C \oplus \delta(\overset{i}{\wedge} T^* \otimes g_1) \otimes C$. Then recall that there are bundles $\mathscr{C}^i$, $i \geq 0$, $\mathscr{L}_j$, $j \geq 1$, and A over B_{H_1} such that $C_0^i \otimes C = f^{-1}\mathscr{C}^i$, $i \geq 0$, $g_j \otimes C = f^{-1}\mathscr{L}_j$, $j \geq 1$, and $T = f^{-1}A$. Then from (7.6) we see that (6.22) has the form

$$\begin{aligned}\mathrm{ch}\, E - \mathrm{ch}\, F &= \sum_{i=1}^{n} (-1)^i \,\mathrm{ch}\, C^i \\ &= \sum_{i=1}^{n} (-1)^i \left\{\mathrm{ch}\,(\overset{i}{\wedge} A^* \otimes \mathscr{C}^0) + \mathrm{ch}\,\delta(\overset{i}{\wedge} A^* \otimes \mathscr{L}_1)\right\}.\end{aligned}$$

The complexification of the δ-sequence

$$\begin{aligned}0 \longrightarrow \mathscr{L}_{i+1} \xrightarrow{\delta} A^* \otimes \mathscr{L}_i \xrightarrow{\delta} \cdots \xrightarrow{\delta} \overset{i-1}{\wedge} A^* \otimes \mathscr{L}_2 \\ \xrightarrow{\delta} \overset{i}{\wedge} A^* \otimes \mathscr{L}_1 \xrightarrow{\delta} \delta(\overset{i}{\wedge} A^* \otimes \mathscr{L}_1) \longrightarrow 0\end{aligned}$$

gives

$$\begin{aligned}\mathrm{ch}\,\delta(\overset{i}{\wedge} A^* \otimes L_1) &= \sum_{k=0}^{i} (-1)^k \,\mathrm{ch}\,(\overset{i-k}{\wedge} A^* \otimes \mathscr{L}_{k+1}) \\ &= \tfrac{1}{2} \sum_{k=0}^{i} (-1)^k \,\mathrm{ch} \overset{i-k}{\wedge} (A^* \otimes C) \cdot \mathrm{ch}\, \mathscr{L}_{k+1},\end{aligned}$$

because $(\overset{*}{\wedge} A^*) \otimes C = \overset{*}{\wedge} (A^* \otimes C)$. Then the topological index $i_t(D + D^*) = 2\chi(M, \Theta) = \chi(M, \Theta_C)$, where

$$0 \longrightarrow \Theta_C \longrightarrow C^0 \otimes C \xrightarrow{D} C^1 \otimes C \xrightarrow{D} \cdots \xrightarrow{D} C^n \otimes C \longrightarrow 0,$$

is given by the formula

$$\begin{aligned}i_t(D + D^*) &= f^{**}\left\{\mathrm{td}\,(A \otimes C) \cdot \frac{\mathrm{ch}\, E - \mathrm{ch}\, F}{e(A)}\right\}[M] \\ &= \frac{1}{2} f^{**}\left\{\frac{\mathrm{td}\,(A \otimes C)}{e(A)} \sum_{i=1}^{n} (-1)^i (\mathrm{ch} \overset{i}{\wedge} (A^* \otimes C) \cdot \mathrm{ch}\, \mathscr{C}^0 \right. \\ &\quad \left. + \sum_{k=0}^{i} (-1)^k \,\mathrm{ch} \overset{i-k}{\wedge} (A^* \otimes C) \cdot \mathrm{ch}\, \mathscr{L}_{k+1})\right\} \qquad (7.15)\end{aligned}$$

If we denote the weights of the real H_1-module V by $x_1, \cdots, x_l$, then

$$\operatorname{td}(A \otimes C) = \prod_{i=1}^{l} \frac{-x_i}{1-e^{x_i}} \frac{x_i}{1-e^{x_i}}, \qquad e(A) = x_1 \cdots x_l .$$

The complexified H_1-module $V \otimes C$ has weights $\pm x_1, \cdots, \pm x_l$, so that

$$\operatorname{ch} \bigwedge^{r} (A^* \otimes C) = \sum_{1 \le i_1 < \cdots < i_r \le l} e^{\pm x_{i_1} \pm \cdots \pm x_{i_r}} .$$

Because $C^0 = A \otimes C$, we get

$$\operatorname{ch} C^0 = \sum_{i=1}^{l} e^{\pm x_i} .$$

Finally if $\pm w_1, \cdots, \pm w_s$ are the weights of the H_1-module L_{k+1}, we have

$$\operatorname{ch} \mathscr{L}_{k+1} = \sum_{i=1}^{s} e^{\pm w_i} ,$$

where the weights w_i are polynomials in the x_i's. See, for example, [4].

Let us write, for a moment, x_i instead $+x_i$, and x_{i+l} instead $-x_i$, for $i = 1, \cdots, l$. Plain substitution of all these expressions into the right hand side of (7.15) shows that there is a polynomial $Q(x_1, \cdots, x_n)$ in the indeterminates $x_1, \cdots, x_n$ such that

$$i_t(D + D^*) = \frac{1}{2} f^{**} \left\{ \prod_{i=1}^{n} \frac{x_i}{1-e^{x_i}} \frac{1}{(x_1 \cdots x_n)^{1/2}} Q(x_1, \cdots, x_n) \right\} [M] .$$

Then from Proposition 6.1 it follows that there are polynomials $Q_k(x_1, \cdots, x_n)$ of degree $k \geq 0$ such that

$$Q(x_1, \cdots, x_n) = (x_1, \cdots, x_n)^{1/2} \sum_{k=0}^{\infty} Q_k(x_1, \cdots, x_n) .$$

Finally, we get

$$(7.16) \qquad i_t(D + D^*) = \frac{1}{2} f^{**} \left\{ \prod_{i=1}^{n} \frac{x_i}{1-e^{x_i}} \sum_{k=0}^{\infty} Q_k(x_1, \cdots, x_n) \right\} [M] .$$

We can write $\prod_{i=1}^{n} \frac{x_i}{1-e^{x_i}} = \sum_{k=0}^{\infty} \phi_k(x)$, where $\phi_k(x)$ is a homogeneous polynomial in the x_i's. Because in formula (7.16) only the term of degree l in the x_i's contributes, we see immediately that

$$(7.17) \qquad i_t(D + D^*) = \tfrac{1}{2} f^{**}\{\phi_0(x)Q_l(x) + \cdots + \phi_l(x)Q_0(x)\}[M] ,$$

writing shortly $Q_s(x)$ instead of $Q_s(x_1, \cdots, x_n)$.

On the other hand, using a special coordinate system so that $X \sim \sum \lambda_i x^i(\partial/\partial x^i)$ up to the higher order at the point $p \in S$, we can write (7.14) in the following form:

$$\chi(M,\Theta) = \sum_{p\in S} \varepsilon_p \frac{1}{(\lambda_1\cdots\lambda_n)^{1/2}} \prod_{i=1}^{n} \frac{\lambda_i}{1-e^{t\lambda_i}} \cdot t^l \sum_{k=0}^{\infty} Q_k(t\lambda_1,\cdots,t\lambda_n)\Big|\text{(the constant term)}.$$

In order to compare the right hand side of the above equation with that of (7.16), the explicit form of the obsolute term is needed.

Because

$$\prod_{i=1}^{n} \frac{\lambda_i}{1-e^{t\lambda_i}} = t^{-n}\phi_0(\lambda) + t^{-n+1}\phi_1(\lambda) + \cdots ,$$

and $Q_k(\lambda) = Q_k(\lambda_1,\cdots,\lambda_n)$ is homogeneous of degree k where $\phi_s(\lambda)$ is a homogeneous polynomial of degree s in the λ's, we get the constant term

$$\chi(M,\Theta) = \sum_{p\in S} \varepsilon_p \frac{1}{(\lambda_1\cdots\lambda_n)^{1/2}}(\phi_0(\lambda)Q_l(\lambda) + \phi_1(\lambda)Q_{l-1}(\lambda) + \cdots + \phi_l(\lambda)Q_0(\lambda)) .$$

Moreover, since $i_t(D + D^*) = 2\chi(M,\Theta)$, we have

$$f^{**}\{\phi_0(x)Q_l(x) + \cdots + \phi_l(x)Q_0(x)\}[M] = \sum_{p\in S} 4\varepsilon_p \frac{1}{(\lambda_1\cdots\lambda_n)^{1/2}}\{\phi_0(\lambda)Q_l(\lambda) + \cdots + \phi_l(\lambda)Q_0(\lambda)\} .$$

Let $\Phi(c_1(T_C(M)),\cdots,c_l(T_C(M)))$, $T_C(M) = T(M)\otimes C$, be the index cocycle (its evaluation on the fundamental cycle $[M]$ is the topological index) of the operator $D + D^*$. With the formal factorization $c(T_C(M)) = \prod_{i=1}^{n}(1 + y_i)$ one can express the Chern classes in the usual way as the elementary symmetric functions in y_i's. Then there is well defined polynomial ϕ such that

$$\phi(y_1,\cdots,y_n) = \Phi(c_1(T_C(M)),\cdots,c_l(T_C(M))) ,$$

so that we can summarize now to obtain

Theorem 7.1. *Let M be a compact $n(=4m)$-dimensional differentiable manifold, and Γ a first order elliptic transitive oriented involutive pseudogroup on M. Let $\Phi(c_1(T_c),\cdots,c_{2m}(T_c))$ be the topological index cocycle of the complex* (2.9) ($k = 1$) *(resolution of the sheaf of germs of Γ-vector fields on M). Let Φ and ϕ be polynomials related as above. Suppose that the Γ-vector field*

X on M has a set S of isolated zero points and that the endomorphism of $T_p(M)$ induced by the Lie derivative L_X has the eigenvalues $\lambda_1(p), \cdots, \lambda_n(p)$. Then

$$\Phi(c_1(T_c), \cdots, c_{2m}(T_c)) = \sum \varepsilon_p \frac{\phi(\lambda_1(p), \cdots, \lambda_n(p))}{(\lambda_1(p) \cdots \lambda_n(p))^{1/2}} ,$$

where $\varepsilon_p = 4 \cdot \operatorname{sgn} \det (1 - e^{L_p})$.

I wish to thank *H*. Goldschmidt who has pointed out to me that recently he has given the most complete proof of the third fundamental theorem for pseudogroups (used in Proposition 2.2), and that the fundamental properties of the brackets (used in § 3) and the proof of Proposition 7.1 can be found in the forthcoming paper of B. Malgrange. Both these papers will appear in the Journal of Differential Geometry.

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